On the Lüroth Quartic Curve.

By Frank Morley.

It has been known since 1870 * that the problem of inscribing a five-line in a planar quartic is poristic; of the ten conditions nine fall on the lines and one on the curve. Thus the quartic is one for which an invariant vanishes, and the degree of this invariant is sought. We use Aronhold's construction of a curve of class 4 from seven given points. And the starting point is the theorem of Prof. Bateman † that the seven points which have the same polar line as to a conic and a cubic give rise to a Lüroth quartic.

For completeness I indicate the proof. A conic and a cubic have the canonical forms (αx^2), (βx^3) where $(x)=0$. The polars of x are (αxy), (βxy^2). Working in a space of three dimensions the line $(y)=0$, $(\alpha xy)=0$ is to touch the quadric (βxy^2). This requires that

$$\Sigma\beta_0\beta_1 x_0 x_1(\alpha_2 x_2-\alpha_3 x_3)^2=0,$$

or

$$(\alpha/\beta)^2/(\alpha^2 x/\beta)=(1/\beta x),$$

and this is a quartic of Lüroth's type. The seven common polar lines are an Aronhold set of double lines of this quartic, and by polarity as to the conic the seven points a_i which have these polar lines are double points of a Lüroth curve of class 4.

§ 1. *The Bateman Conic.*

Take now a conic $(\alpha x)^2$ and a cubic $(\beta x)^3$. The Jacobian of these and a line (ξx)

$$(\alpha x)(\beta x)^2|\alpha\beta\xi|=0$$

gives the net of cubics on the seven points a_i. Referred to one of the points and the corresponding line let the conic be $x_0^2+2x_1x_2$ and the cubic be

$$x_0^3+x_0(\gamma x)_0^2+(\delta x)_0^3.$$

Then for $(\xi x)\equiv x_0$ the Jacobian is

$$(\alpha_1\beta_2-\alpha_2\beta_1)(\alpha x)(\beta x)^2=\beta_2(\beta x)^2x_2-\beta_1(\beta x)^2x_1,$$

so that not only terms in x_0^2 but also the term $x_0x_1x_2$ is missing.

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* Lüroth, *Math. Annalen*, Vol. I.

† American Journal of Mathematics, Vol. XXXVI.

That is, *the seven cubics with double points a_i have their nodal tangents apolar to the conic* α. I will call this conic the Bateman conic.

Given any seven points a_i, cubics on them determine by their remaining intersections a Geiser involution $a^8x^8\xi=0$. If ξ is the join of x and y, then $a^8x^9y=0$, or removing the Jacobian of the net, an a^3x^6, $a^5x^3y=0$. This is the canonic form of the net.*

It may be written as a two-one connex $a^5x^2\xi$, giving for every line ξ the Geiser pair on ξ. This Geiser pair is the neutral pair of the net of binary cubics of ξ, cut out by the net of cubic curves. The quartic, locus of lines ξ for which the Geiser pair come together, is found by making ξ touch the conic $a^5x^2\xi$, and is an $a^{10}\xi^4$.

Write the above two-one connex $a^5x^2\xi$ as $(\gamma x)^2(c\xi)$, and consider $(\gamma x)(\gamma y)\,|cxy|$ an $a^5x^2y^2$. This skew form is the polar conic of y as to its associate cubic, and when y is a_i it is the nodal tangents to the cubic with double point a_i. We have seen that in the case in question these seven line pairs are apolar to a conic. But there are only six independent conics. Thus the required condition is that for arbitrary y the associate conic be apolar to a conic. That is, the six-rowed determinant of all coefficients $\gamma_{ij}c_k$ vanishes. But being a skew determinant it is a square. Thus *a cubic function of the coefficients $\gamma_{ij}c_k$ vanishes.*

§ 2. *The Cubic Invariant of Seven Points.*

For the connex $(\gamma x)^2(c\xi)$ the possible expressions of the third degree are

$$(c\gamma)(c'\gamma')(c''\gamma'')\,|\gamma\gamma'\gamma''|,\quad (c\gamma)(c'\gamma'')(c''\gamma')\,|\gamma\gamma'\gamma''|,$$
$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|,\quad |cc'c''|\,|\gamma\gamma'\gamma''|^2.$$

Of these the first, second, and fourth change sign on interchange of $c'\gamma'$ with $c''\gamma''$, and therefore are zero. Thus the invariant in question is

$$(c\gamma')(c'\gamma'')(c''\gamma)\,|\gamma\gamma'\gamma''|.$$

The invariant expressed in terms of the seven points a_i is of degree 15. But if six points are on a conic it will vanish.

For the form $a^5x^2y^2$ was made up in this way: on the line $\overline{xy}$ is a net of binary cubics, with a neutral pair, and xy are taken harmonic with this neutral pair. If $\overline{ya_i}$ meet the cubic curve with double point at a_i at p_i, then the

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*An expression for the net of cubics on seven given points may be noted, though not of present use. Let A be the Jacobian of cubics on $a_2 \dots . a_7x$, and so on. Then the determinant of seven rows

$$|\,a^2_{i0},\ a^2_{i1},\ a^2_{i2},\ a_{i1}a_{i2},\ a_{i2}a_{i0},\ a_{i0}a_{i1},\ (yDa_i)A_0\,|$$

is the expression in question. For when $x=a_1$, $A_2 \dots . A_7$ vanish and $(yDa_1)A_1$ also vanishes. Presumably this expression for the net is canonic.

neutral pair is a_i and p_i, and if the polar of y as to these be x_i, then the seven points x_i are on the conic $a^5x^2y^2$ associate with y.

If now $a_2 \ldots . a_7$ are on a conic $(\alpha x)^2$, the points $p_2 \ldots . p_7$ are on the line $(\alpha x)(\alpha y)$, and the form $a^5x^2y^2$ becomes $(\alpha x)(\alpha y)|a_1xy|$.

This with a_1 as the reference point $(1, 0, 0)$ and $(\alpha x)^2$ as $x_0^2+2x_1x_2$ becomes

$$(x_0y_0+x_1y_2+x_2y_1)(x_1y_2-x_2y_1),$$

and since there is no term in y_0^2 the invariant of the coefficients vanishes. Thus the expression of degree 15 in a_i breaks up, and removing the factors which vanish when any six of the seven points are on a conic, we are left with *a cubic expression in the* a_i.

Thus, given six of the points, the locus of the seventh, a_7, is a cubic curve. If once more the six are on a conic $(\alpha x)^2$ then the nodal tangents of the cubic $|a_ia_7x|(\alpha x)^2$ are apolar to $(\alpha x)^2$, which is therefore the Bateman conic. Then the tangents at a_7 to the cubic of the system with double point a_7 are apolar to the conic, and this defines the cubic. Thus, if the six points on the conic be given by the binary form $(\beta t)^6$, the locus of a_7 is $(\beta t)^3(\beta t')^3=0$, namely, that cubic on the six points to which the conic (as a line-curve) is apolar.

Thus a special seven-point for which the cubic invariant vanishes is six points on a conic and any point on the apolar cubic through them.

Hence, given any six points $a_1 \ldots . a_6$ we have a counter-six $b_1 \ldots . b_6$ where b_1 is the extra point in which the conic on $a_2 \ldots . a_6$ meets the cubic on $a_1 \ldots . a_6$ and apolar to this conic. The locus of a_7 passes through all twelve points. It is to be noticed that the relations of the points a_i and the points b_i are mutual.

Expressed in terms of Professor Coble's* linear invariants $\bar{a} \ldots . \bar{f}$ of six points, and linear covariants $a \ldots . f$, these being cubics on the points, the covariant cubic in question can be no other than $\Sigma\bar{a}^2a$. This is then an expression for the cubic invariant of seven points.

§ 3. *The Lüroth Invariant.*

If now we map the plane on a cubic surface by means of cubics on the six points $a_1 \ldots . a_6$, the covariant cubic curve becomes a covariant plane of the isolated double-six of lines on the surface. The construction becomes as follows: Let a_i and b_i be a pair of lines of the double-six. Sections of the surface on a_i determine points on b_i. The tangent conic sections determine two points on b_i. There is a conic on the surface through these two points and this determines a point on a_i, on the plane required.

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* *Transactions*, Vol. XVI (1915), § 4.

If from any point where this plane meets the surface we draw the tangent lines we obtain a quartic curve of Lüroth's type. Now a cubic surface has thirty-six double-sixes, and therefore thirty-six such planes. The locus of points on the surface which give rise to Lüroth quartics is then thirty-six planar cubics.

But a covariant of the surface of order 2μ gives an invariant of the corresponding quartic of degree 3μ. Hence, *the Lüroth invariant is of degree* 54.

A line of the surface belongs to sixteen double sixes. Thus the thirty-six planes meet a line of the surface in $16+20$ points. Thus a Lüroth quartic can acquire a double point in two ways. In the one, the lines at the double point are apolar to the points on a double line. In the other the lines at the double point meet the curve again on a line of the curve.

This indicates the nature of the Lüroth invariant I_{54}, namely it is, to the discriminant I_{27} as modulus, the product of two invariants.

Consider a nodal cubic surface in Sylvester's form, $(\varkappa x^3)$ where $(1/\sqrt{\varkappa})=0$. It can be proved that the plane corresponding to the double-six of lines on the node is $(x\sqrt{\varkappa})$.

Hence, when $(1/\sqrt{\varkappa})$ is not 0, there is a covariant of order 16, product of the sixteen planes $(x\sqrt{\varkappa})$, meeting any line of the surface at the sixteen points on it, so that there is an invariant I_{24} which vanishes for a nodal Lüroth quartic of the first kind.

For a nodal Lüroth quartic of the second kind, the inscribed five lines are three lines on the double point, and two lines on a fixed point of the curve. In particular the tangents from the node fall into two sets of three, each set having its contacts on a line. Thus such a quartic is included in those for which three intersections of double lines lie on a line.

Looking then at a double-six from a point y of its cubic surface, the six lines from y to each pair lie on a quadric cone which breaks up into two planes when y is on one of ten planes corresponding to the separation of the six pairs into threes.

If in the case of a nodal cubic $(\varkappa x^3)$ surface these ten planes formed for the nodal double-six have an equation rational in $\varkappa_i$, then an invariant I_{15} vanishes for the nodal Lüroth quartic of the second kind, and the Lüroth invariant will be $I_{27}I_{27}^1+I_{24}I_{15}^2$ where I_{27}^1 is an invariant which is probably the discriminant also.

VINEYARD HAVEN, MASS., *July*, 1918.